

# 12. THE FUNDAMENTAL THEOREM OF ALGEBRA

## §12.1. Extensions of $\mathbb{R}$ and $\mathbb{C}$

If  $F$  is any field (not necessarily a subfield of  $\mathbb{C}$ ) with the property that every non-constant polynomial over  $F$  has a zero in  $F$ , then  $F$  is said to be **algebraically closed**.

### Example 1:

$\mathbb{Q}$  is not algebraically closed since  $x^2 - 2 \in \mathbb{Q}[x]$  but has no zeros in  $\mathbb{Q}$ .

$\mathbb{R}$  is not algebraically closed since  $x^2 + 1 \in \mathbb{R}[x]$  but has no zeros in  $\mathbb{R}$ .

$\mathbb{C}$  is algebraically closed. This is just a restatement of the Fundamental Theorem of Algebra.

We have discussed two types of extensions of fields. At first we only looked within  $\mathbb{C}$ , but, with finite fields, we considered extensions in the sense of quotient rings. We continue that sense here. Within  $\mathbb{C}$ , the following two theorems are obvious, but we'll be showing that there are no extensions *anywhere* with the stated specifications.

**Theorem 1:** There's is no proper extension of  $\mathbb{R}$  of odd dimension over  $\mathbb{R}$ .

**Proof:** Let  $\mathbb{R} < K$  such that  $|K/\mathbb{R}|$  is odd. Let  $\alpha \in K$  such that  $\alpha \notin \mathbb{R}$ .

Then  $\mathbb{R}[\alpha] \leq K$  and  $|K/\mathbb{R}| = |K/\mathbb{R}[\alpha]| \cdot |\mathbb{R}[\alpha]/\mathbb{R}|$  from which we see that  $|\mathbb{R}[\alpha]/\mathbb{R}|$  is odd.

But  $|\mathbb{R}[\alpha]/\mathbb{R}|$  is equal to the degree of  $p(x)$ , the minimum polynomial of  $\alpha$  over  $\mathbb{R}$ .

Since  $p(x)$  has odd degree it has at least one real zero,  $\beta$ . But then  $p(x) = (x - \beta)q(x)$  for some  $q(x) \in \mathbb{R}[x]$ , contradicting the fact that  $p(x)$  is prime over  $\mathbb{R}$ .

**Theorem 2:** There is no extension of  $\mathbb{C}$  of degree 2.

**Proof:** Let  $\mathbb{C} < K$  be such that  $|K/\mathbb{C}| = 2$ .

Let  $\alpha \in K$  such that  $\alpha \notin \mathbb{C}$ . Then  $\mathbb{C}[\alpha] = K$  and the minimum polynomial of  $\alpha$  over  $\mathbb{C}$  has degree 2.

But the zeros of a polynomial over  $\mathbb{C}$  of degree 2 are themselves in  $\mathbb{C}$  by the quadratic equation formula, so  $\alpha \in \mathbb{C}$ , a contradiction.

## §12.2. The Fundamental Theorem of Algebra

Despite its name, the Fundamental Theorem of Algebra is more a theorem of Analysis because it hinges on the continuity properties of the real and complex

numbers as well as on their algebraic properties. But in this proof the analysis is well hidden. Can you find it?

In the following proof we make use of the conjugation automorphism,  $\lambda: \mathbb{C}[x] \rightarrow \mathbb{C}[x]$  where we take the conjugates of the coefficients.

**Example 2:** If  $f(x) = i x^2 + (3 + 4i)x + \omega$ , then  
 $f^\lambda(x) = -i x^2 + (3 - 4i)x + \omega^2$ .

**Theorem 3 (FUNDAMENTAL THEOREM OF ALGEBRA):**

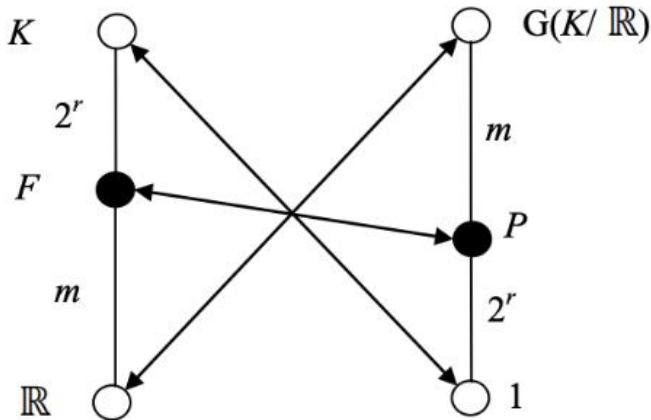
If  $f(x) \in \mathbb{C}[x]$  then  $f(\alpha) = 0$  for some  $\alpha \in \mathbb{C}$ .

**Proof:** Let  $f(x) \in \mathbb{C}[x]$  and let  $g(x) = f(x) f^\lambda(x)$  where  $\lambda$  is the conjugation automorphism. Then  $g(x) \in \mathbb{R}[x]$ .

Let  $K = \mathbb{C}[g(x)]$  and let  $G = G(K/\mathbb{R})$ .

Let  $|G| = 2^r m$  where  $m$  is odd.

The order of a Sylow 2-subgroup,  $P$ , of  $G$  is  $2^r$ .



Let  $F$  be the fixed field of  $P$ .

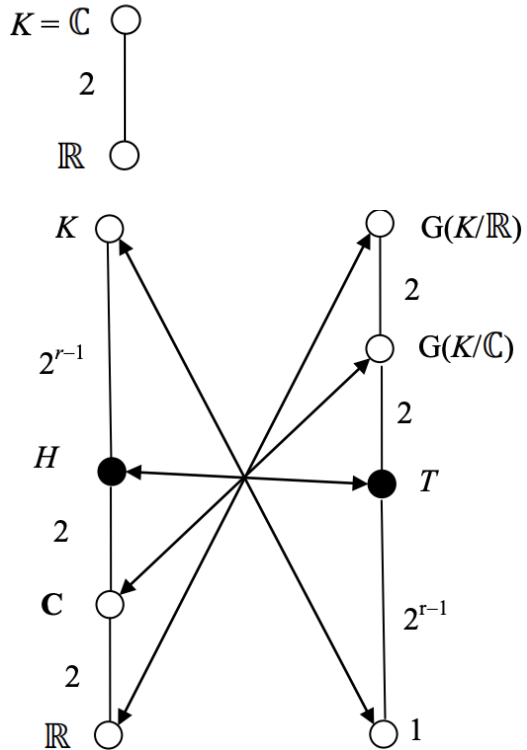
By Theorem 3  $|K/F| = 2^r$  and the fixed field of  $G$  is  $\mathbb{R}$ .

Hence  $|K/\mathbb{R}| = |G| = 2^r m$  and so  $|F/\mathbb{R}| = m$ .

By Theorem 4,  $m = 1$  and  $F = \mathbb{R}$  so  $|K/\mathbb{R}| = |G| = 2^r$ .

Since  $\mathbb{C} \leq K$ ,  $r \geq 1$  and  $|K:\mathbb{C}| = 2^{r-1}$ .

If  $r = 1$ ,  $K = \mathbb{C}$  and so the zeros of  $g(x)$ , and hence those of  $f(x)$ , are already in  $\mathbb{C}$ .



Suppose  $r \geq 2$ . By Theorem 3 the fixed field of  $G(K/\mathbb{C})$  is  $\mathbb{C}$  and  $|G(K/\mathbb{C})| = |K/\mathbb{C}| = 2^{r-1}$ .

Now  $G(K/\mathbb{C})$  has a subgroup  $T$  of order  $2^{r-2}$  and if  $H$  is the fixed field of  $T$ ,  $|K/H| = |P| = 2^{r-2}$ .

Thus  $|H/\mathbb{C}| = 2$  which contradicts Theorem 5.